Closing Tue: TN 4 Closing Thu: TN 5

Entry Tasks (Sigma Notation Practice)

1. Differentiate and integrate:

$$f(x) = \sum_{k=2}^{5} \frac{(-1)^k}{k^3} x^k = \frac{1}{8}x^2 - \frac{1}{27}x^3 + \frac{1}{64}x^4 - \frac{1}{125}x^5$$

$$f'(x) = \sum_{k=2}^{5} \frac{(-1)^k}{k^3} k x^{k-1} = \frac{1}{4}x - \frac{1}{9}x^2 + \frac{1}{16}x^3 - \frac{1}{25}x^4 = \sum_{k=2}^{5} \frac{(-1)^k}{k^2} x^{k-1}$$

$$\int f(x)\,dx = \sum_{k=2}^{5} \frac{(-1)^k}{k^3} \frac{1}{(k+1)} x^{k+1} = \frac{1}{8}\frac{1}{3}x^3 - \frac{1}{27}\frac{1}{4}x^4 + \frac{1}{64}\frac{1}{5}x^5 - \frac{1}{125}\frac{1}{6}x^6$$

2. Combine

$$5\sum_{k=2}^{4} k^2 x^k - 6\sum_{k=2}^{4} \frac{1}{k!} x^k = 5\left(2^2x^2 + 3^2x^3 + 4^2x^4\right) - 6\left(\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4\right)$$

$$= \sum_{k=2}^{4} \left(5k^2 - \frac{6}{k!}\right) x^k$$

TN 5: Using Taylor Series

Here are the 6 series you can quote:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k \;, \qquad \text{for all x}$$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \;, \text{for all x}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \;, \qquad \text{for all x}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \;, \qquad \text{for } -1 < x < 1$$

$$-\ln(1-x) = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} \;, -1 < x < 1$$

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \;, -1 < x < 1$$

Tools for using Taylor Series

1. Substitute (replace x)
2. Integrate

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

3. Differentiate

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

4. Combine

$$\sum_{k=0}^{\infty} k x^k - 3 \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \sum_{k=0}^{\infty} \left(k - \frac{3}{k!}\right) x^k$$

Substitution Questions: Find the Taylor series based at 0, find the first three nonzero terms and give the interval of convergence.

(a) $f(x) = 3e^{2x}$ $= 3\sum_{k=0}^{\infty} \frac{1}{k!}(2x)^k = \sum_{k=0}^{\infty} \frac{3}{k!}(2)^k x^k = 3 + \frac{3}{1!}2^1x^1 + \frac{3}{2!}2^2x^2 + \cdots$

$= 3 + 6x + 6x^2 + \cdots$

For All x $\boxed{-\infty < x < \infty}$

(b) $g(x) = \frac{5}{1-4x}$ $= 5\sum_{k=0}^{\infty}(4x)^k = \sum_{k=0}^{\infty} 5(4)^k x^k = 5 + 5(4)^1x^1 + 5(4)^2x^2 + \cdots$

$= 5 + 20x + 80x^2 + \cdots$

for $-1 < 4x < 1 \Rightarrow \boxed{-1/4 < x < 1/4}$

(c) $h(x) = \frac{3}{2x+1}$ $= 3 \cdot \frac{1}{1-(-2x)} = 3\sum_{k=0}^{\infty}(-2x)^k = \sum_{k=0}^{\infty} 3(-2)^k x^k$

$= 3 + 3(-2)x + 3(-2)^2x^2 + \cdots$

$= 3 - 6x + 12x^2 + \cdots$

for $-1 < -2x < 1 \Rightarrow \boxed{-\frac{1}{2} < x < \frac{1}{2}}$

Combining: Find the Taylor series based at 0, find the first three nonzero terms and give the interval of convergence

(a) $y = 7 + 3x^5 e^{2x}$ $= 7 + 3x^5 \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k = \boxed{7 + \sum_{k=0}^{\infty} \frac{3(2)^k}{k!} x^{k+5}}$

$= \underbrace{7 + 3x^5 + \frac{3(2)}{1!} x^6}_{} + \cdots$

For All x $\boxed{-\infty < x < \infty}$

(b) $y = \frac{5}{1-4x} - \frac{3}{2x+1}$ $= \underbrace{\left(\sum_{k=0}^{\infty} 5(4)^k x^k\right)}_{-1/4 < x < 1/4} - \underbrace{\left(\sum_{k=0}^{\infty} 3(-2)^k x^k\right)}_{-\frac{1}{2} < x < 1/2}$ Combined works for $\boxed{-1/4 < x < 1/4}$

$= \boxed{\sum_{k=0}^{\infty} \left(5 \cdot (4)^k - 3(-2)^k\right) x^k}$

$= \left(5(4)^0 - 3(-2)^0\right) x^0 + \left(5 \cdot 4 - 3(-2)\right) x^1 + \left(5 \cdot 16 - 3 \cdot 4\right) x^2 + \cdots$

$= 2 + (20 + 6) x + (80 - 12) x^2 + \cdots$

$= \underbrace{2 + 26x + 68x^2}_{} + \cdots$

(c) $y = \cos^2(x)$ (Hint: Half-angle)

$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$
$$= \frac{1}{2} + \frac{1}{2}\cos(2x)$$
$$= \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(2x)^{2k}$$
$$= \boxed{\frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2(2k)!} 2^{2k} x^{2k}} = \underbrace{\underbrace{\frac{1}{2}}_{k=0} + \frac{1}{2}}_{1} \underbrace{- \underbrace{\frac{1}{2(2!)}2^2x^2}_{k=1} + \underbrace{\frac{1}{2(4)!}2^4x^4}_{k=2} - \cdots}$$

ONLY CHANGES CONSTANT TERM

CAN LEAVE LIKE THIS

OR WRITE LIKE THIS

SAME

$$= \boxed{1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2(2k)!} 2^{2k} x^{2k}}$$

FOR ALL x

$-\infty < x < \infty$

Integrating Applications

(a) Give the first three nonzero terms of the Taylor Series for

$$\int_0^x 7 + 3t^5e^{2t}dt$$

$$\approx \int_0^x 7 + 3t^5 + 6t^6 + \cdots dt = \int_0^x 7 + \sum_{k=0}^{\infty} \frac{3(2)^k}{k!} t^{k+5} dt$$

$$= 7t + \frac{3}{6}t^6 + \frac{6}{7}t^7 \Big|_0^x \qquad = 7t + \sum_{k=0}^{\infty} \frac{3(2)^k}{k!} \frac{1}{k+6} t^{k+6} \Big|_0^x$$

$$= 7x + \frac{3}{6}x^6 + \frac{6}{7}x^7 + \cdots \qquad = 7x + \sum_{k=0}^{\infty} \frac{3(2)^k}{k!} \frac{1}{(k+6)} x^{k+6}$$

For all x.

(b) Find a Taylor series for:

$$A(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

$$\sin(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \cdots$$

$$\Rightarrow \frac{\sin(t)}{t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k} = 1 - \frac{1}{3!}t^2 + \frac{1}{5!}t^4 - \frac{1}{7!}t^6 + \cdots$$

$$\Rightarrow \int_0^x \frac{\sin(t)}{t}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\frac{1}{(2k+1)} t^{2k+1}\Big|_0^x = t - \frac{1}{3!}\frac{1}{3}t^3 + \frac{1}{5!}\frac{1}{5}t^5 - \cdots\Big|_0^x$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\frac{1}{(2k+1)} x^{2k+1}}$$

For All x.